Extensions on "Sequentially Optimal Mechanisms"

Vasiliki Skreta
UCLA

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Abstract

This document establishes that the result in "Sequentially Optimal Mechanisms" is robust to a number of extensions.

1. Robustness: Longer Horizon & Alternative Degrees of Transparency

We will show that our result is robust in a number of different directions. The first extension considers the possibility that the game lasts arbitrarily long, but finitely many periods. The next three extensions are related to the degree of transparency of mechanisms. We have so far assumed that the seller observes the message that the buyer submits to the mediator, β, the action that he chooses s, and whether trade takes place or not. In the first extension we consider the case where the seller simply observes whether trade took place or not. In the second extension we look at an intermediate case where the seller observes the messages that the buyer submits to the mediator, and whether trade took place or not, but does not observe the action chosen by the buyer. Finally we allow the seller to observe everything, that is the message that the buyer submits to the mediator β, the recommendation that he receives from the mediator n, the action he chooses s and whether trade took place or not. This is the maximal amount of information that the seller can observe.

1.1 Sequentially Optimal Mechanisms for $2 < T < \infty$.

We proceed by induction and we obtain the characterization of sequentially optimal mechanisms for the case that $T > 2$. The overall structure of the proof is as in the two-period case. Of course the execution gets at times more involved and the notation a bit more cumbersome.

Induction Hypothesis: Suppose that we have established that it is optimal to post a price if the game lasts for $T-1$ periods. Then we will establish that the same is true if the game lasts for T periods.

Our initial point is again to establish that it is without any loss to consider the artificial Program B where we have replaced the type space with its convex hull. The analogue of Proposition 3 is :

Proposition A.2 Suppose that the value of Program A and Program B is the same if the game lasts $T-1$ periods, then the value of Program A and Program B is the same if the game lasts T periods.

1.1.1 Revenue Maximizing PBE among 2-Option Mechanisms

We start by looking for a revenue maximizing allocation rule among the allocation rules implemented by a strategy profile of the class *two-options at* $t=1$, *price below the optimal at* $t=2$, and show that a revenue maximizing allocation rule among this class is implemented by a PBE of the game where the seller posts a price in each period. The seller proposes at $t=1$ $M_1=\{(r,z),(1,z_1)\}$, where $r\in[0,1]$ and $z,z_1\in\mathbb{R}$. The buyer's strategy is as follows: types $v\in[a,\bar{v}_1)$ choose (r,z) and types in $(\bar{v}_1,b]$ choose $(1,z_1)$ at $t=1$. Finally type $\bar{v}_1$ is indifferent between choosing: (r,z) at $t=1$ and $(1,z_2)$ at $t=2$ versus choosing $(1,z_1)$ at $t=1$, that is $\bar{v}_1=\frac{z_1-z-(1-r)\delta z_2}{1-r-(1-r)\delta}$ and may be randomizing at $t=1$ between these contracts. By our induction hypothesis at $t=2$ after the history where the buyer chose (r,z) at $t=1$ and no trade took place, the seller chooses a sequence of prices $z_2,z_3,...,z_T$, such that $z_2\leq z_2(\bar{v}_1)$, where $z_2(\bar{v}_1)$ would have been the optimal price given beliefs $F_2(v)=\frac{F(v)}{F(\bar{v}_1)}$. Such an assessment is not necessarily a PBE since the seller after the history where the buyer chose (r,z) at $t=1$, may be choosing a cut-off below the optimal one at $t=2$.

Type $\hat{v}_t=\frac{z+(1-r)\delta^{t-1}z_t-z-(1-r)\delta^t z_{t+1}}{r+(1-r)\delta^{t-1}-r-(1-r)\delta^t}=\frac{z_t-\delta z_{t+1}}{1-\delta}$ is indifferent between choosing (r,z) at $t=1$ and $(1,z_t)$ at t, versus choosing (r,z) at $t=1$, and $(1,z_{t+1})$ at $t+1$.

Sometimes it will be more convenient (but equivalent), to think of the seller as if he is choosing cutoffs $\hat{v}_t$, $t=2,...,T$ then to be choosing prices $z_2,z_3,...,z_T$.

The allocation rules implemented by such strategy profiles are of the form

$$\begin{gathered}
p(v)=r \text{ for } v\in[a,z_T(F_T)) \\
p(v)=r+(1-r)\delta^{T-1} \text{ for } v\in[z_T(F_T),\hat{v}_{T-1}) \\
\ldots\ldots \\
p(v)=r+(1-r)\delta^2 \text{ for } v\in[\hat{v}_3,\hat{v}_2) \qquad , \\
p(v)=r+(1-r)\delta \text{ for } v\in[\hat{v}_2,\bar{v}_1) \\
p(v)=1 \text{ for } v\in[\bar{v}_1,b], \\
\text{for some } \bar{v}_1\in[a,b],\ r\in[0,1],\ z\in\mathbb{R}
\end{gathered} \tag{1}$$

with $\hat{v}_2 \leq \bar{v}_2$, where $\bar{v}_2$ is the optimal cut-off at $t = 2$ given beliefs $F_2(v) = \frac{F(v)}{F(\bar{v}_1)}$, and where $\hat{v}_t$ is the optimal cut-off at t given beliefs $F_t(v) = \frac{F_{t-1}(v)}{F_{t-1}(\hat{v}_{t-1})}$ for $t = 3, ..., T$.

Definition A.1 We call $\mathcal{P}_T^*$ the set of allocation rules that have the shape described in (1) for some $\bar{v}_1 \in [a, b]$, $r \in [0, 1]$, and $\hat{v}_2 \leq v_2(\bar{v}_1)$; where $v_2(\bar{v}_1)$ is the optimal cut-off at $t = 2$ given beliefs $F_2(v) = \frac{F(v)}{F(\bar{v}_1)}$.

We now turn to show that the revenue maximizing element of $\mathcal{P}_T^*$ can be implemented by a PBE of the game where the seller posts a price in each period. For that we use a generalization of Lemma 2, in the main text.

Lemma A.2 Let $\bar{v}_{T-1}$ denote an optimal cut-off at $T-1$ given beliefs $F_{T-1}(v) = \frac{F(v)}{F(\bar{v}_{T-2})}$ then it is increasing in $\bar{v}_{T-2}$.

Proof First recall from Lemma 2 that cut-off $\bar{v}_{T-1}$ determines the optimal price in the final period of the game z_T. At the beginning of $t = T-1$ revenue given $F_{T-1}(v) = \frac{F(v)}{F(\bar{v}_{T-2})}$ can be written as:

$$R_{T-1}(\bar{v}_{T-1}, F_{T-1}) = \frac{1}{F(\bar{v}_{T-2})} \left[\left(F(\bar{v}_{T-2}) - F(\bar{v}_{T-1}) \right) z_{T-1} + \left(F(\bar{v}_{T-1}) - F(z_T) \right) \delta z_T \right].$$

Given that the buyer's strategy is a best response it must hold that $\bar{v}_{T-1} = \frac{z_{T-1} - \delta z_T(\bar{v}_{T-1})}{1-\delta}$. From this we can rewrite $z_{T-1} = (1-\delta)\bar{v}_{T-1} + \delta z_T(\bar{v}_{T-1})$ and substituting this in the expression for revenue we obtain:

$$R_{T-1}(\bar{v}_{T-1}, F_{T-1}) = \frac{1}{F(\bar{v}_{T-2})} \left[\begin{array}{c} \left(F(\bar{v}_{T-2}) - F(\bar{v}_{T-1}) \right) \left((1-\delta)\bar{v}_{T-1} + \delta z_T(\bar{v}_{T-1}) \right) \\ + \left(F(\bar{v}_{T-1}) - F(z_T(\bar{v}_{T-1})) \right) \delta z_T(\bar{v}_{T-1}) \end{array} \right],$$

Since $\frac{1}{F(\bar{v}_{T-2})}$ is a constant $\bar{v}_{T-1}$ maximizes essentially the following expression:

$$\begin{aligned} R_{T-1}(\bar{v}_{T-1}, F_{T-1}) &= \left(F(\bar{v}_{T-2}) - F(\bar{v}_{T-1}) \right) \left((1-\delta)\bar{v}_{T-1} + \delta z_T(\bar{v}_{T-1}) \right) \\ &\quad + \left(F(\bar{v}_{T-1}) - F(z_T(\bar{v}_{T-1})) \right) \delta z_T(\bar{v}_{T-1}) \end{aligned}$$

Now let $\hat{v}_{T-1}$ denote the optimal cut-off at $t = T-1$ given posterior $F_{T-1}(v) = \frac{F(v)}{F(\hat{v}_{T-2})}$ with $\hat{v}_{T-2} > \bar{v}_{T-2}$. From the same arguments as before it follows that $\hat{v}_{T-2}$ maximizes

$$\begin{aligned} R_{T-1}(\hat{v}_{T-1}, \hat{F}_{T-1}) &= \left(F(\hat{v}_{T-2}) - F(\hat{v}_{T-1}) \right) \left((1-\delta)\hat{v}_{T-1} + \delta z_T(\hat{v}_{T-1}) \right) \\ &\quad + \left(F(\hat{v}_{T-1}) - F(z_T(\hat{v}_{T-1})) \right) \delta z_T(\hat{v}_{T-1}), \end{aligned}$$

and since $\hat{v}_{T-2} > \bar{v}_{T-2}$ this expression can be written as

$$\begin{aligned} R_{T-1}(\hat{v}_{T-1}, \hat{F}_{T-1}) &= \left(F(\hat{v}_{T-2}) - F(\bar{v}_{T-2}) \right) \left((1-\delta)\hat{v}_{T-1} + \delta z_T(\hat{v}_{T-1}) \right) \\ &\quad + \left(F(\bar{v}_{T-2}) - F(\hat{v}_{T-1}) \right) \left((1-\delta)\hat{v}_{T-1} + \delta z_T(\hat{v}_{T-1}) \right) \\ &\quad + \left(F(\hat{v}_{T-1}) - F(z_T(\hat{v}_{T-1})) \right) \delta z_T(\hat{v}_{T-1}). \end{aligned}$$

We argue by contradiction. Suppose that $\hat{v}_{T-1} < \bar{v}_{T-1}$ then by Lemma 8 in the main text we also have that $z_T(\hat{v}_{T-1}) \leq z_T(\bar{v}_{T-1})$. From these two observations it follows that

$$\begin{aligned} & (F(\hat{v}_{T-2}) - F(\bar{v}_{T-2}))\left((1-\delta)\hat{v}_{T-1} + \delta z_T(\hat{v}_{T-1})\right) \\ < \quad & (F(\hat{v}_{T-2}) - F(\bar{v}_{T-2}))\left((1-\delta)\bar{v}_{T-1} + \delta z_T(\bar{v}_{T-1})\right). \end{aligned} \tag{2}$$

Also observe that since $\bar{v}_{T-1}$ is the optimal cut-off given beliefs F_{T-1}, then

$$\begin{aligned} & (F(\bar{v}_{T-2}) - F(\hat{v}_{T-1}))\left((1-\delta)\bar{v}_{T-1} + \delta z_T(\bar{v}_{T-1})\right) + (F(\bar{v}_{T-1}) - F(z_T(\bar{v}_{T-1})))\,\delta z_T(\bar{v}_{T-1}), \\ \geq \quad & (F(\bar{v}_{T-2}) - F(\hat{v}_{T-1}))\left((1-\delta)\hat{v}_{T-1} + \delta z_T(\hat{v}_{T-1})\right) + (F(\hat{v}_{T-1}) - F(z_T(\hat{v}_{T-1})))\,\delta z_T(\hat{v}_{T-1}). \end{aligned} \tag{3}$$

Combining (2) and (3) we get that

$$R_{T-1}(\bar{v}_{T-1}, \hat{F}_{T-1}) > R_{T-1}(\hat{v}_{T-1}, \hat{F}_{T-1}),$$

contradicting the optimality of $\hat{v}_{T-1}$. ■

Suppose that for all for $t = \tau, ..., T$, we have demonstrated that $\bar{v}_t$, denotes the optimal cut-off at $t-1$ given beliefs $F_t(v) = \frac{F(v)}{F(\bar{v}_{t-1})}$, is increasing in $\bar{v}_{t-1}$. Then, we can use identical arguments as in Lemma A 2 to establish that:

Lemma A.3 Let $\bar{v}_{t+1}$ denote an optimal cut-off at $t+1$ given beliefs $F_{t+1}(v) = \frac{F(v)}{F(\bar{v}_t)}$ then for $t = 2, ..., T-2$ it is increasing in $\bar{v}_t$.

Proposition A.2 Let p^* denote the solution of $\max_{p \in \mathcal{P}_T^*} R(p)$. Then p^* can be implemented by a PBE of the game where the seller posts a price in each period.

Proof. The seller's can be also written in more traditional way as follows:

$$[1 - F(\bar{v}_1)]z_1 + [F(\bar{v}_1) - F(\bar{v}_2)]\delta z_2 + ... + [F(\bar{v}_{T-1}) - F(z_T)]\delta^{T-1} z_T$$

By recursive substitutions we can write $z_t's$ solely as a function of $v_1, v_2, ..., v_{T-1}$ and r

$$\begin{aligned} z_{T-1} &= (1-\delta)\hat{v}_{T-1} + \delta z_T(\hat{v}_{T-1}) \\ z_{T-2} &= (1-\delta)\hat{v}_{T-2} + \delta\left[(1-\delta)\hat{v}_{T-1} + \delta z_T(\hat{v}_{T-1})\right] \\ z_{T-3} &= (1-\delta)\hat{v}_{T-3} + \delta\left[(1-\delta)\hat{v}_{T-1} + \delta\left((1-\delta)\hat{v}_{T-1} + \delta z_T(\hat{v}_{T-1})\right)\right] \\ & .... \\ z_1 &= \left[1 - r - (1-r)\delta\right]\bar{v}_1 + z + (1-r)\delta\left[(1-\delta)\hat{v}_2 + \delta\left[(1-\delta)\hat{v}_3(\hat{v}_2) + \delta\left[...\right]\right] ....\right]. \end{aligned}$$

Observe that after substituting these cutoffs in the objective function it becomes a linear function of r : the derivative is independent of r. If

$$\frac{\partial R}{\partial r} > 0,$$

then set $r=1$, otherwise set $r=0$.

Now it remains to show that at the optimum the seller will choose $\hat{v}_2=\bar{v}_2$. Note that the choice $\hat{v}_2$ determines the posterior $F_3(v)=\frac{F(v)}{F(\hat{v}_2)}$, which in turn determines F_4 and so on.

Case 1: If $r=1$ then the level of $\hat{v}_2$ is irrelevant since the seller trades with all types with probability 1 at $t=1$.

Case 2: If $r=0$ then, since $\hat{v}_2\leq\bar{v}_2$, by Lemma A.3 we also have that $\hat{v}_t\leq\bar{v}_t$, $t=3,...,T-1$ and $\hat{z}_T\leq z_T$. From these observations we get that

$$\begin{aligned} & [1-F(\bar{v}_1)]\left[(1-r-(1-r)\delta)\,\bar{v}_1+z+(1-r)\delta\left((1-\delta)\bar{v}_2+\delta\left((1-\delta)\bar{v}_3(\bar{v}_2)+\delta\left(\right)...\right)....\right)\right] \\ > \; & [1-F(\bar{v}_1)]\left[(1-r-(1-r)\delta)\,\bar{v}_1+z+(1-r)\delta\left((1-\delta)\hat{v}_2+\delta\left((1-\delta)\hat{v}_3(\hat{v}_2)+\delta\left(\right)...\right)....\right)\right]. \end{aligned} \tag{4}$$

Also since $\bar{v}_2$ solves the seller's problem at the beginning of $t=2$ given beliefs $F_2(v)=\frac{F(v)}{F(\bar{v}_1)}$ we get:

$$\begin{aligned} & \frac{1}{F(\bar{v}_1)}\{[F(\bar{v}_1)-F(\bar{v}_2)]\,z_2+[F(\bar{v}_2)-F(\bar{v}_3)]\,\delta z_3+[F(\bar{v}_3)-F(\bar{v}_4)]\,\delta^2 z_4+ \\ & +...+[F(\bar{v}_{T-2})-F(\bar{v}_{T-1})]\,\delta^{T-3}z_{T-1}+[F(\bar{v}_{T-1})-F(z_{T-1})]\,\delta^{T-2}z_T\} \\ \geq\; & \frac{1}{F(\bar{v}_1)}\{[F(\bar{v}_1)-F(\hat{v}_2)]\,z_2+[F(\hat{v}_2)-F(\hat{v}_3)]\,\delta z_3+[F(\hat{v}_3)-F(\hat{v}_4)]\,\delta^2 z_4 \\ & +...+[F(\hat{v}_{T-2})-F(\hat{v}_{T-1})]\,\delta^{T-3}z_{T-1}+[F(\hat{v}_{T-1})-F(z_{T-1})]\,\delta^{T-2}z_T\}, \end{aligned} \tag{5}$$

but from (4) and (5) we obtain that

$$\begin{aligned} & [1-F(\bar{v}_1)]\left[(1-r-(1-r)\delta)\,\bar{v}_1+z+(1-r)\delta\left((1-\delta)\bar{v}_2+\delta\left((1-\delta)\bar{v}_3(\bar{v}_2)+\delta\left(\right)...\right)....\right)\right] \\ & +...+ \\ & +[F(\bar{v}_{T-1})-F(z_{T-1})]\,\delta^{T-1}z_T \\ > \; & [1-F(\bar{v})]\,[1-F(\bar{v}_1)]\left[(1-r-(1-r)\delta)\,\bar{v}_1+z+(1-r)\delta\left((1-\delta)\hat{v}_2+\delta\left((1-\delta)\hat{v}_3(\hat{v}_2)+\delta\left(\right)...\right)....\right)\right] \\ & +...+ \\ & +[F(\hat{v}_{T-1})-F(z_{T-1})]\,\delta^{T-1}z_T. \end{aligned}$$

Hence at an optimum the seller will set $\hat{v}_2$ equal to its optimal value at $t=2$. From these arguments it follows that the revenue maximizing allocation rule out of $\mathcal{P}_T^*$can be implemented by a PBE of the game where the seller posts a price in each period. ■

Summarizing, if the seller restricts attention to period one mechanisms that contain just two options: one targeted to the "low" types (r,z) and one targeted to the "high" types, then at the optimum this mechanism reduces to a posted price: the options available are $(0,0)$ and $(1,z_1)$. Now we move on to provide necessary conditions that allocation rules satisfy if they are implemented by strategy profiles where the seller employs arbitrary mechanisms at $t = 1$.

1.1.2 The General Case when $2 \leq T < \infty$

As in the case of $T = 2$ we start by "drawing" the allocation from the lower end of types. Let s denote an action that leads to a contract (r, z), that is the smallest probability contract that type a is "choosing" with strictly positive probability at $t = 1$. Also let $[a, \bar{v}_1]$ denote the convex hull of the set of types that choose s, and hence (r, z), with strictly positive probability at $t = 1$. We will use the induction hypothesis to establish necessary conditions that a $PBE-$implementable allocation rule needs to satisfy if the game lasts for T periods.

Definition A.2 An allocation rule is an element of $\mathcal{P}_T$ if it satisfies the following properties (i) increasing in v on $[a, b]$ (ii) $0 \leq p(v) \leq 1$ for $v \in [a, b]$ and (iii)

$$\begin{array}{c} p(v) = r \text{ for } v \in [a, z_T(F_T)) \\ p(v) = r + (1-r)\delta^{T-1} \text{ for } v \in (z_T(F_T), \bar{v}_{T-1}) \\ \ldots\ldots \\ p(v) = r + (1-r)\delta \text{ for } v \in (\bar{v}_2, \bar{v}_1) \\ r + (1-r)\delta \leq p(v) \leq 1 \text{ for } v \in [\bar{v}_1, b] \end{array},$$

for some $\bar{v}_1 \in [a, b]$, $r \in [0, 1]$, $z \in \mathbb{R}$ and $\bar{v}_2$ optimally chosen given some posterior F_2 whose support has convex hull $[a, \bar{v}_1]$, and where $\bar{v}_t$ is optimally chosen given some posterior $F_t(v) = \frac{F_{t-1}(v)}{F_{t-1}(\bar{v}_{t-1})}$ for $t = 3, ..., T$.

We can deduce by the monotonicity of p that its value for types on the boundaries of the various subintervals will be somewhere in between the two steps, for instance $p(z_T(F_T)) \in (r, r + (1-r)\delta^{T-1})$ or $p(\bar{v}_{T-1}) \in (r + (1-r)\delta^{T-1}, r + (1-r)\delta^{T-2})$. It is possible that there exists $z_{\hat{t}}$ and some $\hat{t} = 1, ..., T$ such that $z_{\hat{t}} \leq a$, in which case we have that $p(a) \in [r + (1-r)\delta^{\hat{t}}, r + (1-r)\delta^{\hat{t}-1})$.

Proposition A.3 Let p denote an allocation rule implemented by a PBE of the game, then $p \in \mathcal{P}_T$.

Proof. Consider a PBE assessment (σ, μ) and let p denote the allocation rule implemented by it. Let s denote an action that leads to a contract (r, z). This is the action that leads to the smallest 'r' contract that type a is choosing with positive probability at $t = 1$ or is indifferent between choosing and not choosing. Also let Y denote the subset of $[a, \bar{v}]$ that contains the types of the buyer that report message β and choose s at $t = 1$ with strictly positive probability, and let $[a, \bar{v}]$, with $a \leq \bar{v}$, denote its convex hull. From our induction hypothesis we have that after the history where (r, z) is chosen at $t = 1$ and no trade takes place at $t = 1$, the seller will maximize revenue by posting a price in each period. Let us call this sequence of prices as z_t, $t = 2, ..., T$ and define

$$\begin{aligned} v_L(t) &= \inf \{v \in Y \text{ s.t. } v \text{ accepts } z_t \text{ at } t\} \\ v_H(t) &= \sup \{v \in Y \text{ s.t. } v \text{ accepts } z_t \text{ at } t\} . \end{aligned}$$

By definition types $v_L(t)$ and $v_H(t)$ either choose (r, z) at $t = 1$ and accept z_t at t with positive probability or are indifferent between this sequence of actions and the actions that they are actually choosing. The proof is broken down into four steps.

Step 1: For $v \in (v_L(t), v_H(t))$, where $v_L(t) \neq v_H(t)$ we have that $p(v) = r + (1-r)\delta^{t-1}$. Suppose not, then there exists $v \in (v_L(t), v_H(t))$ such that $p(v) \neq r+(1-r)\delta^{t-1}$, that is it is either a) $p(v) > r+(1-r)\delta^{t-1}$ or b) $p(v) < r+(1-r)\delta^{t-1}$. If $p(v) > r+(1-r)\delta^{t-1}$ then type v must be choosing with positive probability a sequence of actions that implement $\hat{p}, \hat{x}$ such that $\hat{p} > r + (1-r)\delta^{t-1}$. At a PBE the buyer's strategy must be a best response hence it must be the case that $\hat{p}v - \hat{x} \geq (r + (1-r)\delta^{t-1})v - z - (1-r)\delta^{t-1}z_t$. But now since $\hat{p} > r + (1-r)\delta^{t-1}$ it follows that $\hat{p}v_H(t) - \hat{x} > (r + (1-r)\delta^{t-1})v_H(t) - z - (1-r)\delta^{t-1}z_t$, contradicting the fact that $v_H(t)$ chooses (r, z) at $t = 1$, $(0, 0)$ at $t = 2, ..., t-1$ and $(1, z_t)$ at t with positive probability or is indifferent between doing and not doing so. Now if $p(v) < r + (1-r)\delta^{t-1}$ then type v is choosing at $t = 1$ with positive probability a sequence of actions that implement $\hat{p}, \hat{x}$ such that $\hat{p} < r + (1-r)\delta^{t-1}$ and because at a PBE the buyer's strategy is a best response then we have that $\hat{p}v - \hat{x} \geq (r + (1-r)\delta^{t-1})v - z - (1-r)\delta^{t-1}z_t$. But now since $\hat{p} < r + (1-r)\delta^{t}$ and $v_L(t) < v$ it follows that $\hat{p}v_L(t) - \hat{x} > (r + (1-r)\delta^{t-1})v_L(t) - z - (1-r)\delta^{t-1}z_t$, contradicting the fact that $v_L(t)$ chooses (r, z) at $t = 1$, $(0, 0)$ at $t = 2, ..., t-1$ and $(1, z_t)$ at t with positive probability or is indifferent between doing and not doing so.

Step 2: The smallest type that accepts the price at t is indifferent between accepting and rejecting, that is

$$\left(r + (1-r)\delta^{t-1}\right) v_L(t) - \left(z + (1-r)\delta^{t-1}z_t\right) = \left(r + (1-r)\delta^{t}\right) v_L(t) - \left(z + (1-r)\delta^{t}z_{t+1}\right),$$

and for $t = T$ this translates to $z_T = v_L(T)$.

First observe that the fact that at a PBE the buyer's strategy must be a best response to the seller's strategy implies that

$$\left(r + (1-r)\delta^{t-1}\right) v_L(t) - \left(z + (1-r)\delta^{t-1}z_t\right) \geq \left(r + (1-r)\delta^{t}\right) v_L(t) - \left(z + (1-r)\delta^{t}z_{t+1}\right).$$

We now show that it must hold with equality. We argue by contradiction. Suppose not, that is

$$\left(r + (1-r)\delta^{t-1}\right) v_L(t) - \left(z + (1-r)\delta^{t-1}z_t\right) > \left(r + (1-r)\delta^{t}\right) v_L(t) - \left(z + (1-r)\delta^{t}z_{t+1}\right),$$

then the seller can increase z_t by Δz such that

$$\left(r+(1-r)\delta^{t-1}\right)v_L(t)-\left(z+(1-r)\delta^{t-1}z_t\right)-\delta\Delta z=\left(r+(1-r)\delta^{t}\right)v_L(t)-\left(z+(1-r)\delta^{t}z_{t+1}\right),$$

and raise higher revenue at the continuation game that starts at t. All types $v\in(v_L(t),v_H(t))$ still prefer to choose $(1,z_t)$ at t then to choose $(0,0)$. Hence at a PBE we have that

$$\left(r+(1-r)\delta^{t-1}\right)v_L(t)-\left(z+(1-r)\delta^{t-1}z_t\right)=\left(r+(1-r)\delta^{t}\right)v_L(t)-\left(z+(1-r)\delta^{t}z_{t+1}\right). \tag{6}$$

Step 3: For $v<v_L(t)$ we have that $p(v)\leq r+(1-r)\delta^t$, for $t=2,...,T-1$.

We now demonstrate that $p(v)\leq r+(1-r)\delta^t$ for all $v<v_L(t)$. We will argue by contradiction. Suppose that there exists $v<v_L(t)$ such that $p(v)>r+(1-r)\delta^t$. Note that since we are looking at a PBE it must be the case that

$$\begin{aligned} p(v)v-x(v) &\geq \left(\left(r+(1-r)\delta^t\right)\right)v-\left(z+(1-r)\delta^t z_{t+1}\right) \text{ or} \\ \left[p(v)-r-(1-r)\delta^t\right]v &\geq \left(z+(1-r)\delta^t z_{t+1}\right)-x(v). \end{aligned}$$

Now since $v<v_L(t)$ and $p(v)>r+(1-r)\delta^t$ we have that

$$\begin{aligned} \left[p(v)-r-(1-r)\delta^t\right]v_L(t) &> x(v)-z-(1-r)\delta^t z_{t+1} \text{ or} \\ p(v)v_L(t)-x(v) &> \left(\left(r+(1-r)\delta^t\right)\right)v_L(t)-(z+(1-r)\delta^t z_{t+1}) \end{aligned}$$

or by (6)

$$p(v)v_L(t)-x(v)>\left(r+(1-r)\delta^{t-1}\right)v_L(t)-\left(z+(1-r)\delta^{t-1}z_t\right)=p(v_L(t))v_L(t)-x(v_L(t)).$$

But then $v_L(t)$ can benefit by mimicking the behavior of v. Contradiction. Therefore $p(v)\leq r+(1-r)\delta^t$ for all $v<v_L(t)$ and $t=1,...,T-1$. But from Step 1 we know that $p(v)=r+(1-r)\delta^t$ for all $v\in(v_L(t+1),v_H(t+1))$. Now by the monotonicity of p and because $v_L(t)\geq v_H(t+1)$[1] we have that $p(v)=r+(1-r)\delta^t$ for $v\in(v_L(t+1),v_L(t))$.

Step 4: For $v\in(a,v_L(T))$, where $a\neq v_L(T)$ we have that $p(v)=r$. Suppose not, then there exists $v\in(a,v_L(T))$ such that $p(v)\neq r$, that is it is either a) $p(v)>r$ or b) $p(v)<r$. If $p(v)>r$ then type v must be choosing with positive probability a sequence of actions that implement $\hat{p},\hat{x}$ such that $\hat{p}>r$. At

[1] By the monotonicity of p it follows that the smallest type that accepts the price at t is weakly greater than the largest type that accepts the price at $t+1$.

a PBE the buyer's strategy must be a best response hence it must be the case that $\hat{p}v - \hat{x} \geq rv - z$. But now since $\hat{p} > r$ it follows that $\hat{p}v_L(T) - \hat{x} > rv_L(T) - z$, contradicting the fact that $v_L(T)$ chooses (r, z) with positive probability or is indifferent between doing and not doing so. Now if $p(v) < r$ then type v is choosing at $t = 1$ with positive probability a sequence of actions that implement $\hat{p}, \hat{x}$ such that $\hat{p} < r$ and because at a PBE the buyer's strategy is a best response then we have that $\hat{p}v - \hat{x} \geq rv - z$. But now since $\hat{p} < r$ and $a < v$ it follows that $\hat{p}a - \hat{x} > ra - z$, contradicting the fact that a chooses (r, z) at $t = 1$ with positive probability or is indifferent between doing and not doing so.

From the last two steps it follows then that $p(v) = r+(1-r)\delta^t$ for $v \in (v_L(t+1), v_L(t))$, for $t = 1, ..., T-1$ and $p(v) = r$, for $v \in [a, v_L(T))$. So for the case under consideration we have demonstrated that a PBE-implementable allocation rule must belong in the set

$$\begin{gathered} p(v) = r \text{ for } v \in [a, z_T(F_T)) \\ p(v) = r + (1-r)\delta^{T-1} \text{ for } v \in (z_T, \bar{v}_{T-1}) \\ p(v) = r + (1-r)\delta^{T-2} \text{ for } v \in (\bar{v}_{T-1}, \bar{v}_{T-2}) \\ p(v) = r + (1-r)\delta^{T-3} \text{ for } v \in (\bar{v}_{T-2}, \bar{v}_{T-3}) \\ ..... \\ p(v) = r + (1-r)\delta^2 \text{ for } v \in (\bar{v}_3, \bar{v}_2) \\ p(v) = r + (1-r)\delta \text{ for } v \in (\bar{v}_2, \bar{v}_1) \\ r + (1-r)\delta \leq p(v) \leq 1 \text{ for } v \in (\bar{v}_1, b], \\ \text{for some } \bar{v}_1 \in [a, b],\ r \in [0, 1],\ z \in \mathbb{R} \end{gathered} .$$

Note that $p(a)$ cannot be strictly less then r by the definition of (r, z), (in order for $p(a) \leq r$ it must be the case that type a is choosing a sequence of actions that implement $\hat{p} < r$, but this contradicts the definition of (r, z) which is the smallest "r" contract that type a chooses with positive probability at $t = 1$). ■

Corollary 1 *Let $[\hat{v}_L, \hat{v}_H]$ denote the convex hull of types that choose a contract $(\hat{r}, \hat{z})$ at $t = 1$ with positive probability. Then it must be the case that*

$$\begin{gathered} p(v) = \hat{r} \textit{ for } v \in [\hat{v}_L, z_T(F_T)) \\ p(v) = \hat{r} + (1-\hat{r})\delta^{T-1} \textit{ for } v \in (z_T(F_T), \hat{v}_{T-1}) \\ ..... \\ p(v) = \hat{r} + (1-\hat{r})\delta \textit{ for } v \in (\hat{v}_2, \hat{v}_H] \end{gathered} ,$$

and $\hat{v}_2$ optimally chosen given some posterior F_2 whose support has convex hull $[a, \bar{v}_1]$, and where $\hat{v}_t$ is optimally chosen given some posterior $F_t(v) = \frac{F_{t-1}(v)}{F_{t-1}(\hat{v}_{t-1})}$ for $t = 3, ..., T$.

The shape of PBE implementable allocation rules is actually quite unexpected. As in the case of $T = 2$, the shape of allocation rules in $\mathcal{P}_T$ is the same as the one we would get in a scenario where all types in $[a, \bar{v}_1)$ choose a contract (r, z) with probability one at $t = 1$; potentially only the location of the cutoffs differs: if all types in $[a, \bar{v}_1)$ choose (r, z) at $t = 1$ with probability one, then $\bar{v}_2$ must be optimally chosen given beliefs $F_2(v) = \frac{F(v)}{F(\bar{v}_1)}$, whereas now $\bar{v}_2$ must be optimally chosen given some posterior F_2 whose support has convex hull $[a, \bar{v}_1]$.

Our objective is to establish that the best way to separate types in the first period is in two- groups: low and high ones. As in the case of $T = 2$ we show that the seller does not benefit from observing cheap messages and from allowing sophisticated strategies for the buyer.

Can the seller benefit from observing the cheap messages β?

Recall that it is possible that the buyer is reporting message β and then choosing s and reporting $\hat{\beta}$ and then choosing action s. Let $\tilde{F}_2$ denote the seller's posterior after he observes (β, s) and let $\hat{F}_2$ denote the seller's posterior after she observes $(\hat{\beta}, s)$. Suppose that F_2 denotes the seller's posterior after she observes only action s. Our objective is to compare $\bar{v}_2(F_2)$ with $\bar{v}_2(\hat{F}_2)$ and $\bar{v}_2(\tilde{F}_2)$. First let us examine how $\bar{v}_2(\hat{F}_2)$ and $\bar{v}_2(\tilde{F}_2)$ relate to each other.

Lemma A.4 Consider a PBE and let $\bar{v}_2(\tilde{F}_2)$, respectively $\bar{v}_2(\hat{F}_2)$, denote the cutoffs that the seller will choose at $t = 2$ after a history where she observed β and s, and $\hat{\beta}$ and s respectively. Then it must be the case that $\bar{v}_2(\tilde{F}_2) = \bar{v}_2(\hat{F}_2)$.

Proof. Suppose not, and without any loss let $\bar{v}_2(\tilde{F}_2) < \bar{v}_2(\hat{F}_2)$. Now since $\bar{v}_2(\tilde{F}_2) = \frac{z_2 - \delta z_3(\bar{v}_2(\tilde{F}_2))}{1-\delta}$ and z_3 is increasing in $\bar{v}_2$, we have that $z_2(\tilde{F}_2) < z_2(\hat{F}_2)$. Then for all $v \in V$ it holds that

$$[r(s) + (1 - r(s))\delta]\, v - \left[z - (1 - r(s))\delta z_2(\tilde{F}_2)\right] > [r(s) + (1 - r(s))\delta]\, v - \left[z - (1 - r(s))\delta z_2(\hat{F}_2)\right],$$

hence for all $v \in [z_2(\tilde{F}_2), b]$ the buyer strictly prefers to report β instead of $\hat{\beta}$, at least for the portion of the time that those types plan to chose s. But then when the seller sees $\hat{\beta}$ and s, she can infer that the valuation of the buyer is below $z_2(\tilde{F}_2)$, which in turn implies that a price of $z_2(\hat{F}_2) > z_2(\tilde{F}_2)$ cannot be optimal. Contradiction. Hence given some mechanism and a communication strategy the choice of s uniquely determines the optimal price at $t = T$. ■

Now we turn to investigate the relationship of $\bar{v}_2(F_2)$ with $\bar{v}_2(\hat{F}_2)$ and $\bar{v}_2(\tilde{F}_2)$. Using a procedure identical to the one employed to prove Lemma 4 in the main text we obtain that:

Lemma A.5 "Cheap" information in $\beta's$ does not lead to higher prices at $t = 2$; the cutoff $\bar{v}_2(F_2) \geq \bar{v}_2(\tilde{F}_2) = \bar{v}_2(\hat{F}_2)$.

Using Lemma A.5 one can establish an analogue Lemma 5 which states that it is without any loss to view mechanisms to a set of contracts.

Does the seller benefit from "sophisticated" strategies of the buyer?

In order to investigate this question, we need to find out which types may be choosing a contract (r, z) with positive probability, ((r, z) is again the smallest "r" contract that type a is choosing with strictly positive probability at $t = 1$, or is indifferent between choosing or not). As in the case of $T = 2$ we establish that only types in $[\bar{v}_2, \bar{v}_1]$ may be choosing a contract other than (r, z).

Lemma A.6 Consider a PBE where $[a, \bar{v}_1]$, denotes the convex hull of the set of types that choose (r, z) with positive probability at $t = 1$. Also let $\bar{v}_2$ denote the cut-off that the seller will chose at $t = 2$ after the history that the buyer chose (r, z) at $t = 1$ and no trade took place. Then only types in $[\bar{v}_2, \bar{v}_1]$ may be choosing a contract different from (r, z) with positive probability at $t = 1$.

Proof. We will argue by contradiction. Suppose that there exist $v \in [a, \bar{v}_2)$ choosing a contract $(\hat{r}, \hat{z})$ different from (r, z) with positive probability at $t = 1$.

Claim 1: The convex hull of the set of types that choose $(\hat{r}, \hat{z})$ at $t = 1$ cannot be a singleton.

If there is just one type, call it $\hat{v} \in [a, \bar{v}_1)$, choosing contract $(\hat{r}, \hat{z})$ with positive probability at $t = 1$, then it must be the case that when the seller observes $(\hat{r}, \hat{z})$ chosen at $t = 1$ and no trade taking place, then she can figure out that the valuation of the buyer is equal to $\hat{v}$ and hence she will post a price equal to $\hat{v}$ and the buyer will accept. In other words we will have that

$$p(\hat{v})\hat{v} - x(\hat{v}) = (\hat{r} + (1 - \hat{r})\delta)\hat{v} - (z + (1 - r)\delta\hat{v}) = \hat{r}v - \hat{z}.$$

We will show that this is impossible. From the fact that (r, z) is the smallest r contract that type a is choosing with positive probability at $t = 1$ we have that $\hat{r} > r$. Otherwise type a would have a profitable deviation. To see this, note that because type $\hat{v}$ is choosing contract $(\hat{r}, \hat{z})$ we have that

$$\hat{r}\hat{v} - \hat{z} \geq p(a)\hat{v} - x(a).$$

If $p(a) = r$ and $\hat{r} < r$ then

$$\hat{r}a - \hat{z} > p(a)a - x(a).$$

If $p(a) = r + (1 - r)\delta^{t-1}$, for some t, (which arises is $z_t < a$ for all $t = 1, ..., T$) then it must then be the case that $\hat{r} \geq r + (1 - r)\delta^t$, otherwise, that is if $\hat{r} < r + (1 - r)\delta^t$ type a would have a profitable deviation. We have therefore demonstrated that $\hat{r} > r$. Now from Proposition 9 we know that $p(v) \leq r + (1 - r)\delta$ for $v \in [a, \bar{v}_1)$ and from the previous observation we have that $r + (1 - r)\delta < \hat{r} + (1 - \hat{r})\delta$. This together with the monotonicity of p which imply that $\hat{v}$ cannot be an element of $[a, \bar{v}_1)$.

Claim 2: Let $(\hat{v}_L, \hat{v}_H)$ denote the convex hull of the set of types that choose contract $(\hat{r}, \hat{z})$ with positive probability. Then we will show that it must be the case that $(\hat{v}_L, \hat{v}_H)$ must be contained in one of the subintervals $(\hat{v}_L, \hat{v}_H) \subset (z_T, \bar{v}_{T-1})$ or $(\hat{v}_L, \hat{v}_H) \subset (\bar{v}_t, \bar{v}_{t-1})$, for some $t = 2, ..., T-1$.

Our objective is to show that the only equilibrium feasible case is when either $(\hat{v}_L, \hat{v}_H) \subset (z_T, \bar{v}_{T-1})$ or $(\hat{v}_L, \hat{v}_H) \subset (\bar{v}_t, \bar{v}_{t-1})$ holds. Suppose not, then it must be the case that $(\hat{v}_L, \hat{v}_H)$ has a non-empty intersection with two consecutive intervals, say $(\bar{v}_{t+1}, \bar{v}_t)$ and $(\bar{v}_t, \bar{v}_{t-1})$, but then by Proposition 9 and Corollary 1 we know that it must be the case

$$\begin{aligned} p(v) &= r + (1-r)\delta^t \text{ for } v \in (\hat{v}_L, \hat{v}_H) \cap (\bar{v}_{t+1}, \bar{v}_t) \text{ and} \\ p(v) &= r + (1-r)\delta^{t-1} \text{ for } v \in (\hat{v}_L, \hat{v}_H) \cap (\bar{v}_t, \bar{v}_{t-1}), \end{aligned} \tag{7}$$

but since some of these types are choosing a contract $(\hat{r}, \hat{z})$ at $t = 1$ with positive probability, then it must also be the case that either

$$\begin{aligned} p(v) &= \hat{r} + (1-\hat{r})\delta^{\hat{t}} \text{ for } v \in (\hat{v}_L, \hat{v}_H) \cap (\bar{v}_{t+1}, \bar{v}_t) \text{ and} \\ p(v) &= \hat{r} + (1-\hat{r})\delta^{\hat{t}-1} \text{ for } v \in (\hat{v}_L, \hat{v}_H) \cap (\bar{v}_t, \bar{v}_{t-1}), \end{aligned} \tag{8}$$

for some $\hat{t} = 1, ..., T-1$, or

$$\begin{aligned} p(v) &= \hat{r} \text{ for } v \in (\hat{v}_L, \hat{v}_H) \cap (\bar{v}_{t+1}, \bar{v}_t) \text{ and} \\ p(v) &= \hat{r} + (1-\hat{r})\delta^{T-1} \text{ for } v \in (\hat{v}_L, \hat{v}_H) \cap (\bar{v}_t, \bar{v}_{t-1}) \end{aligned} \tag{9}$$

Now combining (7) and (8) we have that the following must be true

$$\begin{aligned} r + (1-r)\delta^{t-1} &= \hat{r} + (1-\hat{r})\delta^{\hat{t}-1} \\ r + (1-r)\delta^t &= \hat{r} + (1-\hat{r})\delta^{\hat{t}} \end{aligned}$$

from the first equality we have that

$$\begin{aligned} r(1-\delta^{t-1}) &= \hat{r}(1-\delta^{\hat{t}-1}) + \delta^{\hat{t}-1} \\ r(1-\delta^{t-1}) &= \hat{r}(1-\delta^{\hat{t}-1}) + \delta^{\hat{t}-1} - \delta^{t-1} \\ r &= \frac{\hat{r}(1-\delta^{\hat{t}-1}) + \delta^{\hat{t}-1} - \delta^{t-1}}{(1-\delta^{t-1})}, \end{aligned}$$

now substituting this expression in the second equality we get that

$$\begin{aligned} \frac{\hat{r}(1-\delta^{\hat{t}-1}) + \delta^{\hat{t}-1} - \delta^{t-1}}{(1-\delta^{t-1})} + \left(\frac{(1-\delta^{t-1}) - \hat{r}(1-\delta^{\hat{t}-1}) - \delta^{\hat{t}-1} + \delta^{t-1}}{(1-\delta^{t-1})}\right)\delta^t &= \hat{r} + (1-\hat{r})\delta^{\hat{t}} \\ \frac{\hat{r}(1-\delta^{\hat{t}-1})(1-\delta^t)}{(1-\delta^{t-1})} + \frac{\delta^{\hat{t}-1} - \delta^{t-1}}{(1-\delta^{t-1})} + \left(\frac{(1-\delta^{t-1}) - \delta^{\hat{t}-1} + \delta^{t-1}}{(1-\delta^{t-1})}\right)\delta^t &= \hat{r}(1-\delta^{\hat{t}}) + \delta^{\hat{t}} \end{aligned}$$

$$\frac{\hat{r}(1-\delta^{\hat{t}-1})(1-\delta^{t})}{(1-\delta^{t-1})} - \hat{r}(1-\delta^{\hat{t}}) = \delta^{\hat{t}} - \frac{\delta^{\hat{t}-1}-\delta^{t-1}}{(1-\delta^{t-1})} - \left(\frac{(1-\delta^{t-1})-\delta^{\hat{t}-1}+\delta^{t-1}}{(1-\delta^{t-1})}\right)\delta^{t}$$

$$\begin{aligned}
&\frac{\hat{r}\left(1-\delta^{t}-\delta^{\hat{t}-1}+\delta^{\hat{t}-1}\delta^{t}-1+\delta^{\hat{t}}+\delta^{t-1}-\delta^{t-1}\delta^{\hat{t}}\right)}{(1-\delta^{t-1})} \\
= \;&\frac{\delta^{\hat{t}}-\delta^{\hat{t}}\delta^{t-1}-\delta^{\hat{t}-1}+\delta^{t-1}-\delta^{t}+\delta^{t}\delta^{t-1}+\delta^{t}\delta^{\hat{t}-1}-\delta^{t-1}\delta^{t}}{(1-\delta^{t-1})}
\end{aligned}$$

$$\begin{aligned}
\frac{\hat{r}\left(-\delta^{t}-\delta^{\hat{t}-1}+\delta^{\hat{t}}+\delta^{t-1}\right)}{(1-\delta^{t-1})} &= \frac{\delta^{\hat{t}}-\delta^{\hat{t}-1}+\delta^{t-1}-\delta^{t}}{(1-\delta^{t-1})} \\
\hat{r} &= \frac{\delta^{\hat{t}}-\delta^{\hat{t}-1}+\delta^{t-1}-\delta^{t}}{-\delta^{t}-\delta^{\hat{t}-1}+\delta^{\hat{t}}+\delta^{t-1}} = 1,
\end{aligned}$$

hence the desired condition holds only for $\hat{r} = 1-$ using reverse steps we can also show that in order that the desired equalities hold it must be the case that

$$r = 1,$$

but then $r = \hat{r}$ contradicting the supposition that types in $(\hat{v}_L, \hat{v}_H)$ choose a contract $(\hat{r}, \hat{z})$ (different from contract (r, z)) with positive probability.

Now let us examine the other possibility. From (7) and (9) it follows that we must have

$$\begin{aligned}
\hat{r} + (1-\hat{r})\delta^{T-1} &= r + (1-r)\delta^{t-1} \text{ and} \\
\hat{r} &= r + (1-r)\delta^{t}
\end{aligned}$$

Substituting the second expression into the first we get that

$$\begin{aligned}
r + (1-r)\delta^{t} + (1-r-(1-r)\delta^{t})\delta^{T-1} &= r + (1-r)\delta^{t-1} \\
-r\delta^{t} + r\delta^{t-1} - r\delta^{T-1} + r\delta^{t}\delta^{T-1} &= \delta^{t-1} - \delta^{t} - \delta^{T-1} + \delta^{t}\delta^{T-1} \\
r &= \frac{\delta^{t-1} - \delta^{t} - \delta^{T-1} + \delta^{t}\delta^{T-1}}{\delta^{t-1} - \delta^{t} - \delta^{T-1} + \delta^{t}\delta^{T-1}} = 1,
\end{aligned}$$

which is impossible for the same reasons as before. Hence the only feasible scenario is that the convex hull of the set of types that are choosing some contract $(\hat{r}, \hat{z})$ with positive probability must be either $(\hat{v}_L, \hat{v}_H) \subset (z_T, \bar{v}_{T-1})$ or $(\hat{v}_L, \hat{v}_H) \subset (\bar{v}_t, \bar{v}_{t-1})$, which completes what we wanted to show.

Claim 3: Claim 2 is impossible.

From Claim 2 we know that all the types that choose $\hat{r}$ with positive probability must be contained in $(\bar{v}_t, \bar{v}_{t-1})$. Then it must be the case that the price that the seller will post at $t = 2$ after the history that the buyer chose $(\hat{r}, \hat{z})$ at $t = 1$ and no trade took place must be in $(\bar{v}_t, \bar{v}_{t-1})$, but then $\hat{p} = \hat{r} + (1 - \hat{r})\delta$, which is impossible for the reasons explained in the proof of Claim 1.

Hence only types in $(v_2, \bar{v}_1)$ may be choosing some contract other than contract (r, z) with positive probability, the reason why this is possible for those types is because we have no restrictions on the shape of the allocation rule for types in $[\bar{v}_1, b]$. ■

Let $m(v)$ denote the probability that type v is choosing a contract (r, z). We assume that m is a measurable function of v. A consequence of Lemma A.6 is that even if we allow for any possible randomization, the posterior at $t = 2$ after the buyer chose (r, z) at $t = 1$, is going to be of the form

$$F_2^m(v) = \begin{cases} \frac{F(v)}{F(\bar{v}_2) + \int_{\bar{v}_2}^{\bar{v}_1} m(s) dF(s)}, & v \in [a, \bar{v}_2) \\ \frac{F(\bar{v}_2) + \int_{\bar{v}_2}^{v} m(s) dF(s)}{F(\bar{v}_2) + \int_{\bar{v}_2}^{\bar{v}_1} m(s) dF(s)}, & v \in [\bar{v}_2, \bar{v}_1], \end{cases} \tag{10}$$

when $\bar{v}_2 > 0$. Suppose that the posterior at $t = 2$ is given by (10), $m(s) \in [0, 1]$, and let $\bar{v}_2^m$ denote an optimal cut-off at $t = 2$ given beliefs $F_2^m(v)$. Also let $\bar{v}_2$ denote an optimal cut-off at $t = 2$ given beliefs $F_2(v) = \frac{F(v)}{F(\bar{v}_1)}$. The result that follows states that $\bar{v}_2 \geq \bar{v}_2^m$. The reason for this is that from Lemma A.6 we know that only types above $\bar{v}_2$ may be actually choosing some contact other than (r, z) with positive probability.

Lemma A.7 $\bar{v}_2 \geq \bar{v}_2^m$.

Proof. We argue by contradiction. Suppose that $\bar{v}_2 < \bar{v}_2^m$. From Lemmata A.2 and A.3 it follows that

$$\bar{v}_3(\bar{v}_2) \leq \bar{v}_3(\bar{v}_2^m); ..; \bar{v}_{T-1}(\bar{v}_{T-2}) \leq \bar{v}_{T-1}(\bar{v}_{T-2}^m); z_T(v_{T-1}) \leq z_T(v_{T-1}^m)$$

Since $\bar{v}_2$ is the optimal cut-off given beliefs $F_2(v) = \frac{F(v)}{F(\bar{v}_1)}$, the difference in expected revenue with cut-off $\bar{v}_2$ and cut-off $\bar{v}_2^m$ is positive. Using the observations above we get that

$$\begin{aligned} & (1-\delta)\frac{1}{F(\bar{v}_1)}\left[\int_{\bar{v}_2}^{\bar{v}_2^m} s dF(s) + \int_{\bar{v}_2}^{\bar{v}_2^m} F(t)dt - \int_{\bar{v}_2}^{\bar{v}_2^m} F(\bar{v}_1)dt\right] \\ & + \frac{1}{F(\bar{v}_1)}\Sigma_{t=3}^{T-1}(\delta^{t-1} - \delta^t)\left[\int_{v_t(\bar{v}_{t-1})}^{v_t(\bar{v}_{t-1}^m)} s dF(s) + \int_{v_t(\bar{v}_{t-1})}^{v_t(\bar{v}_{t-1}^m)} F(t)dt - \int_{v_t(\bar{v}_{t-1})}^{v_t(\bar{v}_{t-1}^m)} F(\bar{v}_1)dt\right] \\ & + \frac{1}{F(\bar{v}_1)}\delta\left[\int_{z_T(\bar{v}_{T-1})}^{z_T(\bar{v}_{T-1}^m)} s dF(s) + \int_{z_T(\bar{v}_{T-1})}^{z_T(\bar{v}_{T-1}^m)} F(t)dt - \int_{z_T(\bar{v}_{T-1})}^{z_T(\bar{v}_{T-1}^m)} F(\bar{v}_1)dt\right] \\ \geq\ & 0 \end{aligned}$$

where we can just ignore the multiplication of every term by the positive constant $\frac{1}{F(\bar{v}_1)}$. Because

$$F(\bar{v}_1) > \left(F(\bar{v}_2) + \int_{\bar{v}_2}^{\bar{v}_1} m(s)dF(s)\right)$$

we obtain

$$\begin{aligned}
&(1-\delta)\left[\int_{\bar{v}_2}^{\bar{v}_2^m} sdF(s) + \int_{\bar{v}_2}^{\bar{v}_2^m} F(t)dt - \int_{\bar{v}_2}^{\bar{v}_2^m}\left(F(\bar{v}_2) + \int_{\bar{v}_2}^{\bar{v}_1} m(s)dF(s)\right)dt\right] \\
&+\Sigma_{t=3}^{T-1}(\delta^{t-1}-\delta^t)\left[\int_{v_t(\bar{v}_{t-1})}^{v_t(\bar{v}_{t-1}^m)} sdF(s) + \int_{v_t(\bar{v}_{t-1})}^{v_t(\bar{v}_{t-1}^m)} F(t)dt - \int_{v_t(\bar{v}_{t-1})}^{v_t(\bar{v}_{t-1}^m)}\left(F(\bar{v}_2) + \int_{\bar{v}_2}^{\bar{v}_1} m(s)dF(s)\right)dt\right] \\
&+\delta\left[\int_{z_T(\bar{v}_{T-1})}^{z_T(\bar{v}_{T-1}^m)} sdF(s) + \int_{z_T(\bar{v}_{T-1})}^{z_T(\bar{v}_{T-1}^m)} F(t)dt - \int_{z_T(\bar{v}_{T-1})}^{z_T(\bar{v}_{T-1}^m)}\left(F(\bar{v}_2) + \int_{\bar{v}_2}^{\bar{v}_1} m(s)dF(s)\right)dt\right] \\
> \;& 0
\end{aligned}$$

contradicting the optimality $\bar{v}_2^m$. ■

Given Lemmata A.5 and A.7 the result follows exactly as in Theorem 1.

Theorem A.3 Suppose that $T < \infty$. Then, under non-commitment the seller maximizes expected revenue by posting a price in each period.

Proof. The result can be established following the exact lines of the proof of Proposition 5. For each allocation rule in $\mathcal{P}_T$ we can construct an allocation rule in $\mathcal{P}_T^*$ that generates higher expected revenue for the seller. As in the proof of Proposition 5 we ignore all sequential rationality constraints and chose p in the range $[\bar{v}_1, b]$ optimally, respecting only the requirement that p is monotonic. We get that

$$\begin{aligned}
\hat{p}(v) &= r + (1-r)\delta \text{ for } v \in [\bar{v}_1, v^{**}) \\
\hat{p}(v) &= 1 \text{ for } v \in [v^{**}, b],
\end{aligned}$$

where v^{**} is given by (11, main text). From Lemmata A.5 and A.7 we have that $v_2 \le \bar{v}_2(\bar{v}_1)$ and from Lemma A.3 we have that $v_2(\bar{v}_1) \le v_2(v^{**})$. From the last two inequalities we get that $v_2 \le v_2(v^{**})$. The optimal allocation rule is an element of $\mathcal{P}_T^*$. From Proposition A.2 we know that the revenue maximizing allocation rule is implemented by a PBE of the game where the seller posts a price in each period. ■

1.2 Alternative Degrees of Transparency

In the next three units we establish that our result is robust to a number of alternative assumptions regarding the degree of transparency of mechanisms.

1.3 Sequentially Optimal Mechanisms with Minimal Amount of Information

Suppose that all that the seller observes is whether the buyer obtained the object or not. Then at $t = 2$ after the history where no trade took place at $t = 1$ the seller's beliefs will be the same irrespective of the actions and the exchange of messages that took place at $t = 1$. We show that if an allocation rule is implemented by a PBE of the game where the seller simply observes whether trade took place or not, then it can be written as a linear combination of allocation rules in (12) in the main text. We sketch the main two steps required to establish this.

Let $[a, \bar{v}]$, with $a < \bar{v}$, denote the convex hull of the set of types that at $t = 1$ choose with positive probability actions that lead with strictly positive probability to no-trade.[2] Those types choose actions that lead to contracts of the following form: $(r_1, z_1), (r_2, z_2), (r_3, z_3), ..., (r_k, z_k)$, where $r_i < 1$, $i = 1, .., k$.[3] Since the seller does not observe any of this and the only information she obtains is whether trade took place or not, she will post the same price at $t = 2$ irrespective of the actions and the messages chosen by the buyer. Let $\hat{z}$ denote the price that the seller will post at $t = 2$, after the history of no trade at $t = 1$. Now the fact that at a PBE the buyer's strategy has to be a best response at each node implies that types above $\hat{z}$ will be accepting this price at $t = 2$. For $v \in [\hat{z}, \bar{v}]$ we must then have that $p(v) = r_i + (1 - r_i)\delta = r_i + \delta - \delta r_i = (1 - \delta) r_i + \delta$, which is increasing in r_i. Hence by the monotonicity of p we have that higher types are choosing higher probability contracts at $t = 1$. This in turn implies that the seller's beliefs assign weakly less weight to types closer to $\bar{v}$ then to types closer to $\hat{z}$. This observation is formalized in the proof of Lemma A.1, that follows, where we establish that $\hat{z} \leq z_2(\bar{v})$, (again $z_2(\bar{v})$ is the price that will be optimal if the posterior is given by $F_2(v) = \frac{F(v)}{F(\bar{v})}$.) With some abuse of notation let $r(v)$, $z(v)$ denote the contract that is chosen by type v at $t = 1$. Then after the history where no trade took place at $t = 1$ the seller's beliefs at $T = 2$ are given by $F_2(v) = \frac{\int_a^v (1-r(s))dF(s)}{\int_a^{\bar{v}} (1-r(s))dF(s)}$, where $\int_a^{\bar{v}} (1 - r(s))dF(s) > 0$ because $a < \bar{v}$ and $(1 - r(s)) > 0$ for all $s \in [a, \bar{v}]$.

Lemma A.1 Let $\hat{z}$ denote the optimal price at $T = 2$ given beliefs $F_2(v) = \frac{\int_a^v (1-r(s))dF(s)}{\int_a^{\bar{v}} (1-r(s))dF(s)}$.Then we have that $\hat{z}_2 \geq z_2(\bar{v})$.

Proof. The price at $t = 2$ is given by

$$z_2 \equiv \inf \left\{ v \in [a, \bar{v}] \text{ such that } \int_v^{\tilde{v}} t dF_2(t) - \int_v^{\tilde{v}} [1 - F_2(t)] dt \geq 0, \text{ for all } \tilde{v} \in [v, b] \right\}.$$

For $F_2 = \frac{\int_a^v (1-r(s))dF(s)}{\int_a^{\bar{v}} (1-r(s))dF(s)}$ where $\int_a^{\bar{v}} (1 - r(s))dF(s) > 0$, the expression $\int_v^{\tilde{v}} t dF_2(t) - \int_v^{\tilde{v}} [1 - F_2(t)] dt$ can be

[2] If $a = \bar{v}$ then the seller's problem at $t = 2$ is trivial: she will post a price equal to $\bar{v}$.

[3] We assume a countable number of actions for simplicity. Nothing depends on this simplification.

rewritten as:

$$\begin{aligned}
& \frac{1}{\int_a^{\bar{v}}(1-r(s))dF(s)}\left[\int_v^{\tilde{v}} t(1-r(t))dF(t) - \int_v^{\tilde{v}}\left(\int_a^{\bar{v}}(1-r(s))dF(s) - \int_a^{t}(1-r(s))dF(s)\right)dt\right] \\
= & \frac{1}{\int_a^{\bar{v}}(1-r(s))dF(s)}\left[\int_v^{\tilde{v}} t(1-r(t))dF(t) - \int_v^{\tilde{v}}\left(\int_t^{\bar{v}}(1-r(s))dF(s)\right)dt\right]
\end{aligned}$$

and z_2 can be equivalently be defined as

$$z_2 \equiv \inf\left\{\begin{array}{c} v\in[a,\bar{v}] \text{ such that } \int_v^{\tilde{v}} t(1-r(t))dF(t) - \int_v^{\tilde{v}}\left(\int_t^{\bar{v}}(1-r(s))dF(s)\right)dt \geq 0, \\ \text{for all } \tilde{v}\in[v,b] \end{array}\right\}.$$

Our objective is to establish that $\hat{z} \leq z_2(\bar{v})$. We will argue by contradiction. Suppose that $\hat{z} > z_2(\bar{v})$, then by the definition of $\hat{z}$ it follows that there exists $\tilde{v} \in [z_2(\bar{v}), b]$ such that

$$0 > \int_{z_2(\bar{v})}^{\tilde{v}} t(1-r(t))dF(t) - \int_{z_2(\bar{v})}^{\tilde{v}}\left(\int_t^{\bar{v}}(1-r(s))dF(s)\right)dt,$$

because r is increasing in s we have that

$$\begin{aligned}
0 &> \int_{z_2(\bar{v})}^{\tilde{v}} t(1-r(t))dF(t) - \int_{z_2(\bar{v})}^{\tilde{v}}\left(\int_t^{\bar{v}}(1-r(s))dF(s)\right)dt \qquad (11)\\
&\geq \int_{z_2(\bar{v})}^{\tilde{v}} t(1-r(t))dF(t) - \int_{z_2(\bar{v})}^{\tilde{v}}\left(\int_t^{\bar{v}}(1-r(t))dF(s)\right)dt.
\end{aligned}$$

Now we will show that we can break the interval $[z_2(\bar{v}), \tilde{v}]$ into subintervals of types that choose actions that lead to the same contracts at $t = 1$. We do this by establishing that the set of types that choose actions that lead to the same contract is convex. In particular we show that if $(\underline{v}, \bar{v})$ is the convex hull of the set of types that choose the same action at $t = 1$, than types in $(\underline{v}, \bar{v})$ can be only randomizing at $t = 1$ among actions that lead to the same contract. If a type in $(\underline{v}, \bar{v})$ is randomizing between different sequences of actions it must be the case that $p = \hat{p}$ and $x = \hat{x}$. Recall that after the history of no trade the seller posts a price $\hat{z}$ at $t = 2$. Then best response constraints at $t = 2$ imply that types above $\hat{z}$ are accepting the price that the seller posts at $t = 2$. This in turn implies that if a type above $\hat{z}$ is randomizing among actions that lead to different contracts, then it must be the case that $p = \tilde{p}$, which implies that $(1-\delta)r_i + \delta = (1-\delta)\tilde{r}_i + \delta$ and $x = \tilde{x}$ which implies that $z_i + (1-r_i)\delta\hat{z} = \tilde{z}_i + (1-\tilde{r}_i)\delta\hat{z}$. But then from the last two observations we have that $r_i = \tilde{r}_i$ and $z_i = \tilde{z}_i$, which is clearly the same contract. Now for types below $\hat{z}$ if there are randomizing between different actions at $t = 1$ it immediately follows that $r_i = \tilde{r}_i$ and $z_i = \tilde{z}_i$. Hence the buyer can be only randomizing among actions that lead to the same contract.

Suppose that types in subinterval $[z_2(\bar{v}), \hat{v}_1]$ choose actions that lead to some contract (r_1, z_1), types in subinterval $[\hat{v}_1, \hat{v}_2]$ choose actions that lead to contract (r_2, z_2) and so forth. Given this observation the right hand side of (11) can be rewritten as

$$\begin{aligned}
&\int_{z_2(\bar{v})}^{\hat{v}_1} t(1-r_1)dF(t) - \int_{z_2(\bar{v})}^{\hat{v}_1} \left(\int_t^{\bar{v}} (1-r_1)dF(s) \right) dt \\
&+ \int_{\hat{v}_1}^{\hat{v}_2} t(1-r_2)dF(t) - \int_{\hat{v}_1}^{\hat{v}_2} \left(\int_t^{\bar{v}} (1-r_2)dF(s) \right) dt + \\
&+ .... + \\
&+ \int_{\hat{v}_{k-1}}^{\hat{v}_k} t(1-r_k)dF(t) - \int_{\hat{v}_{k-1}}^{\hat{v}_k} \left(\int_t^{\bar{v}} (1-r_k)dF(s) \right) dt.
\end{aligned}$$

Now by the definition of $z_2(\bar{v})$ we have that

$$\begin{aligned}
&\int_{z_2(\bar{v})}^{\hat{v}_1} t(1-r_1)dF(t) - \int_{z_2(\bar{v})}^{\hat{v}_1} \left(\int_t^{\bar{v}} (1-r_1)dF(s) \right) dt \\
= \; & (1-r_1) \left[\int_{z_2(\bar{v})}^{\hat{v}_1} tdF(t) - \int_{z_2(\bar{v})}^{\hat{v}_1} \left(\int_t^{\bar{v}} dF(s) \right) dt \right] \geq 0.
\end{aligned}$$

If

$$\int_{\hat{v}_1}^{\hat{v}_2} t(1-r_2)dF(t) - \int_{\hat{v}_1}^{\hat{v}_2} \left(\int_t^{\bar{v}} (1-r_2)dF(s) \right) dt \geq 0$$

then we have that

$$\begin{aligned}
&\int_{z_2(\bar{v})}^{\hat{v}_1} t(1-r_1)dF(t) - \int_{z_2(\bar{v})}^{\hat{v}_1} \left(\int_t^{\bar{v}} (1-r_1)dF(s) \right) dt \\
&+ \int_{\hat{v}_1}^{\hat{v}_2} t(1-r_2)dF(t) - \int_{\hat{v}_1}^{\hat{v}_2} \left(\int_t^{\bar{v}} (1-r_2)dF(s) \right) dt \geq 0,
\end{aligned}$$

otherwise, that is if

$$\int_{\hat{v}_1}^{\hat{v}_2} t(1-r_2)dF(t) - \int_{\hat{v}_1}^{\hat{v}_2} \left(\int_t^{\bar{v}} (1-r_2)dF(s) \right) dt < 0$$

then we have that

$$\begin{aligned}
& \int_{z_2(\bar{v})}^{\hat{v}_1} t(1-r_1)dF(t) - \int_{z_2(\bar{v})}^{\hat{v}_1} \left(\int_t^{\bar{v}} (1-r_1)dF(s)\right) dt \\
& + \int_{\hat{v}_1}^{\hat{v}_2} t(1-r_2)dF(t) - \int_{\hat{v}_1}^{\hat{v}_2} \left(\int_t^{\bar{v}} (1-r_2)dF(s)\right) dt \\
\geq \; & \int_{z_2(\bar{v})}^{\hat{v}_1} t(1-r_2)dF(t) - \int_{z_2(\bar{v})}^{\hat{v}_1} \left(\int_t^{\bar{v}} (1-r_2)dF(s)\right) dt \\
& + \int_{\hat{v}_1}^{\hat{v}_2} t(1-r_2)dF(t) - \int_{\hat{v}_1}^{\hat{v}_2} \left(\int_t^{\bar{v}} (1-r_2)dF(s)\right) dt;
\end{aligned}$$

this inequality is due to $(1-r_2) \leq (1-r_1)$, which follows by the monotonicity of r, (see main text), and the fact that $\int_{z_2(\bar{v})}^{\hat{v}_1} t(1-r_1)dF(t) - \int_{z_2(\bar{v})}^{\hat{v}_1} \left(\int_t^{\bar{v}} (1-r_1)dF(s)\right) dt \geq 0$, but then again by the definition of $z_2(\bar{v})$ it follows that

$$\begin{aligned}
& \int_{z_2(\bar{v})}^{\hat{v}_1} t(1-r_2)dF(t) - \int_{z_2(\bar{v})}^{\hat{v}_1} \left(\int_t^{\bar{v}} (1-r_2)dF(s)\right) dt \\
& + \int_{\hat{v}_1}^{\hat{v}_2} t(1-r_2)dF(t) - \int_{\hat{v}_1}^{\hat{v}_2} \left(\int_t^{\bar{v}} (1-r_2)dF(s)\right) dt \geq 0.
\end{aligned}$$

Continuing in a similar fashion we can show that the right hand side of (11) is greater than zero. Contradiction. We have therefore established that $\hat{z} \leq z_2(\bar{v})$. ■

Now, given Lemma A.1 , we now argue, somewhat informally, that allocation rules implemented by an assessment where the seller obtains minimal amount of information, can be written as a linear combination of allocation rules in (12) in the main text. Let r_L, z_L and r_H, z_H denote respectively the smaller and the larger probability contracts that are chosen with strictly positive probability at $t = 1$ by types in $[a, \bar{v}]$. Since the seller in the scenario under consideration merely observes whether trade took place or not, then the price at $t = 2$ is independent of the buyer's choice at $t = 1$. From Lemma A.1 we have that $\hat{z} \leq z_2(\bar{v})$. Then an allocation rule implemented by a PBE where the seller observes only whether trade took place at $t = 1$ or not, can be written as a linear combination of the following two allocation rules:

$$\begin{aligned}
p(v) &= r_L \text{ for } v \in [a, \hat{z}) \\
p(v) &= r_L + (1-r_L) \text{ for } v \in [\hat{z}, \bar{v}) \\
p(v) &= 1 \text{ for } v \in [\bar{v}, b]
\end{aligned}$$

and

$$
\begin{aligned}
p(v) &= r_H \text{ for } v \in [a, \hat{z}) \\
p(v) &= r_H + (1 - r_H) \text{ for } v \in [\hat{z}, \bar{v}) \\
p(v) &= 1 \text{ for } v \in [\bar{v}, b].
\end{aligned}
$$

We call the set of allocation rules implemented by $PBE's$ where the seller no information $\bar{\mathcal{P}}_2$. Take an element of $\bar{\mathcal{P}}_2$, call it $\bar{p}$. Since $\bar{p}$ is a linear combination of elements[4] of $\mathcal{P}_2$, call them p_i, $i = 1, ..., n$ and because expected revenue R is linear in p, (see (8), main text), it can be written as

$$R(\bar{p}) = R(\Sigma_{i=1}^{L} \alpha_i p_i) = \Sigma_{i=1}^{L} \alpha_i R(p_i).$$

Now we know by the proof of Theorem 1, main text that each element of $\mathcal{P}_2$ is dominated in terms of expected revenue by an element of $\mathcal{P}_2^*$. Let us call p_i^* the element of $\mathcal{P}_2^*$ that dominates p_i, then we have that

$$R(\bar{p}) = \Sigma_{i=1}^{L} \alpha_i R(p_i) \leq \Sigma_{i=1}^{L} \alpha_i R(p_i^*) \leq R(p^*).$$

where $p^* \in \arg\max_{i \in \{1,...,n\}} R(p_i^*)$. It follows that each element of $\bar{\mathcal{P}}_2$ is dominated in terms of expected revenue by an element of $\mathcal{P}_2^*$.

Theorem A.2 Suppose that the seller observes only whether trade takes place or not. Then under non-commitment the seller maximizes expected revenue by posting a price in each period.

1.4 Intermediate Amount of Information: Seller observes only "cheap" messages and trade/no trade

Here we look at an intermediate case where the seller observes the messages that the buyer submits to the mediator and whether trade took place or, but does not observe the action chosen by the buyer. In this environment where the interests of the buyer and the seller are directly opposite, allowing the seller to observe apart from whether trade took place or not, the cheap messages submitted by the buyer is redundant. Let's see why. Suppose that the seller observes the reports the buyer sends to the mediator as well as whether trade took place or not. Based on the information that the seller observes at $t = 1$ she will post a price at $t = 2$. Given that the "cheap" message of the buyer at $t = 1$ influences only the seller's beliefs at $t = 2$ and nothing at $t = 1$, all types of the buyer prefer the message that will lead to the lowest price at $t = 2$. Hence all types will then choose the same "cheap" message, which implies that the possibility that the seller observes the cheap messages, on top of whether trade takes place or not, does not add any

[4] We assume that they are finitely many for simplicity.

information. What about now if the seller observes the recommendation that the buyer receives from the mediator? Again as before $n's$ may play the role of coordination to particular continuation equilibria. Each β induces a probability distribution over $n's$ and each of these $n's$ coordinates to a particular $t = 2$ price. It follows that all types prefer to choose the message β that leads to the lowest expected $t = 2$ price. Hence still all types of the buyer choose the same β.

1.5 Sequentially Optimal Mechanisms with Maximal Amount of Information

What is a revenue maximizing PBE if we allow the seller to observe also the recommendation that the buyer receives from the mediator, n? The first step in finding an optimal allocation is to derive the set of $PBE-$implementable allocation rules. In order to do so, we investigate the role of the recommendations that the buyer receives from the mediator. We argue that $n's$ play a role of a coordination device to a particular "continuation" equilibrium. A continuation equilibrium in this setup is an assessment, that conditional on a given message submitted by the buyer to the mediator, and on the recommendation that the buyer receives from the mediator, satisfies the requirements of PBE. We illustrate this role of $n's$ with an example.

Example 1 *Consider a PBE where the mechanism that the seller employs at $t = 1$ consists of a game form that contains two actions s and $\hat{s}$ such that when s is chosen the contract is (r, z) and when $\hat{s}$ is chosen the contract is $(\hat{r}, \hat{z})$. These two contracts are such that the following inequalities are true*

$$r < \hat{r} < r + (1 - r)\delta < \hat{r} + (1 - \hat{r})\delta.$$

In order to complete the description of a mechanism let us describe the mediator: it allows the report of a single message β. Given β it sends recommendation n_1 with probability a half and recommendation n_2 with probability a half. There are many possible allocation rules implemented by "continuation equilibria" where the seller employs this game form at $t = 1$. Two possibilities are as follows: 1) all types below $\bar{v} = \frac{\hat{z}+(1-\hat{r})\delta\hat{z}_2 - z-(1-r)\delta z_2(\bar{v})}{\hat{r}+(1-\hat{r})\delta-(r+(1-r)\delta)}$ choose (r, z) and all types above $\bar{v}$ choose $(\hat{r}, \hat{z})$ at $t = 1$ and 2) all types below $\tilde{v} = \frac{\hat{z}+(1-\hat{r})\delta\hat{z}_2 - z-(1-r)\delta z_2(\tilde{v})}{\hat{r}+(1-\hat{r})\delta-(r+(1-r)\delta)}$ choose (r, z) and all types above $\tilde{v}$ choose $(\hat{r}, \hat{z})$ at $t = 1$, where we take $\bar{v} < \tilde{v}$. Then consider an assessment where when the recommendation is n_1 the buyer's strategy is such that types above $\bar{v}$ choose $(\hat{r}, \hat{z})$ at $t = 1$ and types below $\bar{v}$ choose (r, z) at $t = 1$, whereas when the recommendation is n_2 the buyer's strategy is such that types above $\tilde{v}$ choose $(\hat{r}, \hat{z})$ at $t = 1$ and types below $\tilde{v}$ choose (r, z) at $t = 1$. Then, from the ex-ante point of view the allocation rule will be of the form $p^{(\beta,n_1)}$ times the *probability that the buyer will receive recommendation* n_1*, where*

$$\begin{aligned} p^{(\beta,n_1)}(v) &= r \textit{ for } v \in [a, z_2(\bar{v})) \\ p^{(\beta,n_1)}(v) &= r + (1-r)\delta \textit{ for } v \in [z_2(\bar{v}), \bar{v}) \\ p^{(\beta,n_1)}(v) &= \hat{r} + (1-\hat{r})\delta \textit{ for } v \in [\bar{v}, b], \end{aligned}$$

and it will be of the form $\mathrm{p}^{(\beta,n_2)}$ *times the probability that the buyer will receive recommendation* n_2

$$\begin{aligned} p^{(\beta,n_2)}(v) &= r \textit{ for } v \in [a, z_2(\tilde{v})) \\ p^{(\beta,n_2)}(v) &= r + (1-r)\delta \textit{ for } v \in [z_2(\tilde{v}), \tilde{v}) \\ p^{(\beta,n_2)}(v) &= \hat{r} + (1-\hat{r})\delta \textit{ for } v \in [\tilde{v}, b]. \end{aligned}$$

This example, though admittedly simplistic, demonstrates the role of $n's$ as a coordination device on a particular continuation equilibrium. It also illustrates, that it is possible that given a message that the buyer submits to the mediator, β, a given action chosen by the buyer s is followed by different prices at $t = 2$ depending on the recommendation that buyer received by the mediator. That is, it is possible that $z_2(\tilde{v}) > z_2(\bar{v})$. This is contrary to Lemma 4, in the main text, which says that if the seller does *not* observe n, it cannot be the case that depending on the message that the buyer submitted to the mediator, a given action of the buyer at $t = 1$ is followed by *different* prices at $t = 2$.

In general, matters are quite more complicated since the message that the buyer submits to the mediator β influences the probability distribution over the recommendations that he receives, and consequently the continuation allocations. Before we proceed to describe necessary conditions that allocation rules satisfy if they are implemented by assessments where the seller observes the recommendations by the mediator, we would like to note that allowing for the possibility that the seller observes the recommendations of the mediator has the same effect as allowing the seller to submit messages to the mediator, if these messages can be observed by the buyer; they both function as a way to coordinate on a specific "continuation equilibrium."

Proposition A.1 Suppose that $T = 2$ and that the seller observes the message that the buyer sends to the mediator, the recommendation that the buyer receives from \ the mediator, the action he chooses, and whether trade took place or not. Then, if an allocation rule is implemented by an assessment that is a PBE, it can be written as a linear combination of allocation rules described in (12) in the main text.

Proof. A given report by the buyer to the mediator β induces a probability distribution over recommendations $n's$. The recommendations $n's$ play a role as a belief's coordination device: each pair of cheap signals (β_i, n_j) determines a continuation assessment and an allocation rule that has the shape of the ones in (12), main text. When the valuation of the buyer is v then, the probability that $p^{(\beta_i,n_j)}(v)$ is the relevant allocation rule is given by the probability that type v will report message β times the probability that the mediator will send recommendation n given message β, and $p(v)$ is then the expectation with respect to all pairs (β, n). It follows that in the case that the seller observes the recommendations that the buyer receives by the mediator, the allocation rule is a linear combination of allocation rules in (12) in the main text. ■

But again from Proposition 4 we have that a revenue maximizing element of $\mathcal{P}_2^*$ is implemented by a PBE of the game where the seller posts a price in each period. We have therefore demonstrated that:

Theorem A.1 Suppose that $T = 2$ and that the seller observes the message that the buyer sends to the mediator, the recommendation that the buyer receives from the mediator, the action he chooses, and whether trade took place or not. Then under non-commitment the seller maximizes expected revenue by posting a price in each period.

Let us conclude our exploration of the various environments where our result is robust with a final remark. Given that the seller employs deterministic game forms all the earlier analysis goes through assuming that the seller instead of actions, the $s's$, observes contracts (r, z).